[2] Steele, F. A., "The Optical Characteristics of Paper," *Paper Trade Journal*, Vol. 100, No. 37, March 21, 1935, pp. 37–42.

[3] Judd, D. B., *Color in Business, Science, and Industry*, Wiley, New York, 1952, pp. 314–350.

[4] Berg, P. W. and McGregor, J. L., *Elementary Partial Differential Equations*, Holden-Day, San Francisco, 1966, pp. 96–110.

An Approximate Distribution of Particle Mass Flux in a High Altitude Solid Propellant Rocket Plume

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Nomenclature

c = constant in mass flux equation
F = motor thrust
$g(r, \phi)$ = radial mass flux
$h(\phi), h(\xi)$ = angular dependent part of radial mass flux
I_{sp} = motor specific impulse
k = constant in fractional mass flow equation
R_e = nozzle exit radius
r = radial coordinate
$W(\phi)$ = particle mass flow within the conical angle ϕ
W_p = total particle mass flow
$w(\phi), w(\xi)$ = fractional particle mass flow $[= W(\phi)/W_p]$
x_p = mass fraction of metal oxide particle in exhaust
ϕ = angular coordinate
ϕ_m = limiting particle streamline angle
ξ = nondimensional angular coordinate $(= \phi/\phi_m)$

I. Introduction

SOLID propellant rocket motors with thrusts from 100 to 10,000 lbf are used in space applications such as staging and orbit changes. When metalized propellants are used the exhaust products normally contain about 30% (by weight) metal oxide particles. Recurring problems with motors of this type are caused by the heating, pressures, and contamination on adjacent surfaces which are struck by the particles in the exhaust plume. Surface contamination is a particularly important problem because of the sensitivity to contamination of thermal control surfaces and solar cells. Because of this sensitivity, contamination by the particles remains a significant problem for separation distances far beyond those at which heating and pressures are no longer important.

Analyses which assess these effects must utilize some distribution of particle mass in the plume. Although there are computational codes[1,2] available which predict particle trajectories and densities in plumes, it would be useful to have some simple, but realistic and moderately accurate, analytical specification of the particle mass flux. The value of such a model is suggested by the widely used Hill-Draper[3] model for a gaseous vacuum plume. In fact, Hill and Draper's method of fitting an analytical approximation to detailed numerical gas dynamic plume solutions can also be applied to particle plumes. This Note presents a simple analytical model for the particle mass flux in a solid propellant motor plume which is derived in this manner.

II. Derivation

Detailed method-of-characteristics solutions for two-phase high-altitude plumes show that far from the nozzle exit the particle trajectories become straight and the particle plume resembles a spherical source flow. For these conditions the particle mass flow contained within a conical particle stream surface (Fig. 1) with half-angle ϕ is

$$W(\phi) = 2\pi \int_o^\phi r^2 g(r, \phi) \sin\phi \, d\phi \tag{1}$$

If ϕ_m is the limiting particle streamline which contains essentially all the particle mass flow, conservation of particle mass requires

$$W(\phi_m) = W_p = x_p(F/I_{sp}) \tag{2}$$

For a sourcelike flow the angular and radial dependence of the particle mass flux are separable:

$$g(r, \phi) = cW_p h(\phi)/r^2 \tag{3}$$

With this assumed form for the particle mass flux and the nondimensional variables $\xi = \phi/\phi_m$ and $w(\xi) = W(\phi)/W_p$, Eq. (1) becomes

$$w(\xi) = 2\pi c\phi_m \int_o^\xi h(\xi) \sin(\xi\phi_m) \, d\xi \tag{4}$$

Boundary values which must be imposed are $w(0) = 0$; $w(1) = 1$; $h(1) = 0$; and the constant c will be chosen so that $h(0) = 1$.

The quantity of primary interest is the angular dependent mass flux $h(\xi)$, which will be chosen to match detailed method-of-characteristics solutions. However, the procedure to be used

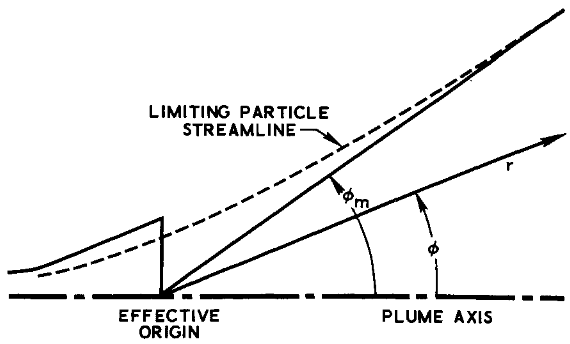

Fig. 1 Plume geometry and coordinate system.

TWO PHASE METHOD OF CHARACTERISTICS SOLUTIONS			
THRUST, lb_f	r/R_e	ϕ_m, deg	SYMBOL
2,000	26.8	35	○
2,000	80	35	△
70,000	13.5	40	◊
	$w(\xi) = \mathrm{erf}(k\xi^2)$		——

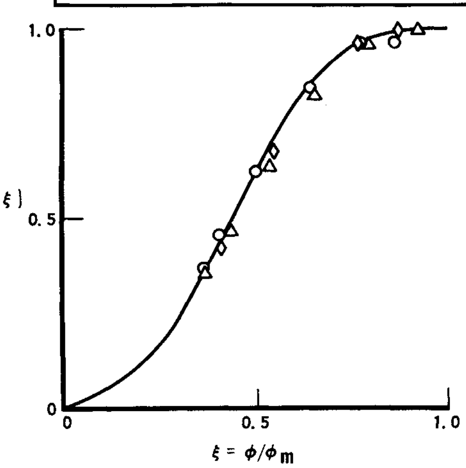

Fig. 2 Normalized particle fractional mass flow.

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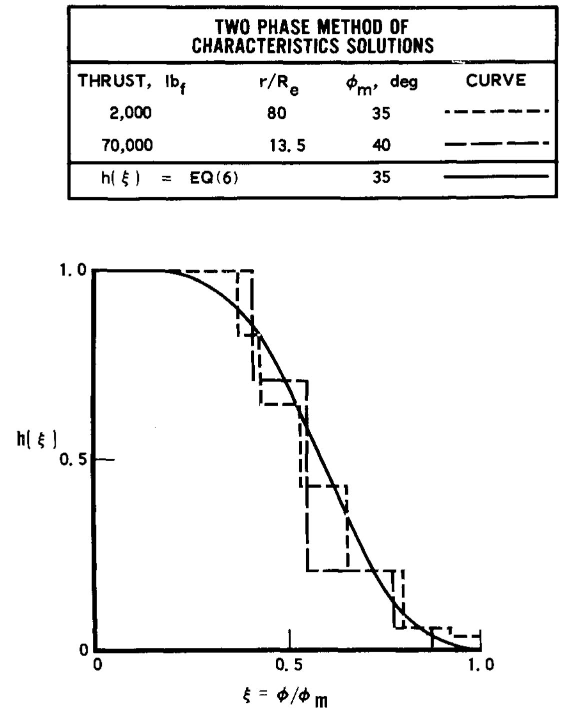

Fig. 3 Normalized particle mass flux distribution.

will fit an analytical expression to $w(\xi)$, the fractional mass flow, rather than to $h(\xi)$ itself. $h(\xi)$ is then obtained by solving Eq. (4). This approach was taken because the numerical solutions available for comparison allowed $w(\xi)$ to be obtained more accurately and with less effort than $h(\xi)$. In addition, matching $w(\xi)$ directly insures that continuity of particle mass is automatically satisfied at each angular position.

The primary source of data for the comparison is from the detailed two-phase plume solutions for two motors.[4] These motors had thrusts of 2000 and 70,000 lbf and mass-mean particle radii of 1.75 and 4.0 μm, respectively. Both had conical nozzles with 15° half-angles and area ratios of 13. The data for only these two motors are used in the subsequent comparisons, but similar, detailed solutions for motors with different thrusts, particle sizes, area ratios, and nozzle contours have been examined and yield essentially the same results.

The fractional particle mass flow at different axial locations for the two plumes are shown in Fig. 2. An analytical function which closely matches the numerical solutions is

$$w(\xi) = \operatorname{erf}(k\xi^2) \tag{5}$$

The constant k must be chosen both to fit the data and to satisfy the boundary condition $w(1) = 1$ with sufficient accuracy. A value $k = 2.5$ meets these conditions. The boundary condition $w(0) = 0$ is satisfied identically. Since the error function is itself defined as an integral, Eq. (4) can be solved simply by comparing integrands and the result is†

$$h(\xi) = \phi_m \xi e^{-k^2\xi^4}/\sin(\xi\phi_m) \tag{6}$$

The mass flux at any point in the plume is then given by

$$g(r, \phi) = 2kW_p \xi e^{-k^2\xi^4}/\pi^{3/2}\phi_m r^2 \sin(\xi\phi_m) \tag{7}$$

The maximum mass flux is on the plume axis and is

$$g(r, 0) = 2kW_p/\pi^{3/2}\phi_m{}^2 r^2 \tag{8}$$

A qualitative comparison between the normalized angular dependent mass flux given by Eq. (6) and discrete distributions given by the numerical solution[4] is made in Fig. 3. The agreement appears satisfactory for both motors. In addition, the particle mass flux on the plume axis predicted by Eq. (8) is compared with the numerical solutions in Fig. 4. At sufficiently large distances the analytical solution underpredicts the numerical results by less than 10%.

III. Discussion and Conclusions

The quantities needed in order to apply the model are the location of the effective origin of the particle source flow and the limiting particle streamline angle ϕ_m, in addition to the normal motor parameters. For the plumes examined in this study the effective origin for the particle source flow was essentially at the exit plane and it is suggested that this location is adequate for most applications. Fortunately, at sufficiently large distances the mass flux is very insensitive to the choice of effective origin. For the several numerical solutions examined, the limiting particle streamline angle ϕ_m varied between about 35° and 40°, depending on motor size and the smallest particle size used in the calculations. In general, the limiting streamline angle increases as the particle size decreases and for a fixed particle size, increases with increasing motor size. Since the fraction of the total particle mass contributed by the smaller particles decreases as the motor size increases, these two effects act to cancel and the effective limiting streamline (that which contains essentially all the particle mass) is not particularly sensitive to motor size. For this reason it is suggested that a nominal value of ϕ_m near 35° or 40° is sufficiently accurate for most applications. For example, a common method of estimating the particle mass flux has been to assume that it is uniformly distributed within the conical streamline with half-angle equal to that of the nozzle wall at the exit plane. For a conical nozzle with an angle of 15°, the mass flux using this estimate is nearly twice that predicted by the more accurate approximation given by Eq. (7). The present model therefore offers substantially improved estimates even if the limiting particle streamline angle ϕ_m is not accurately established.

The axial location at which the particle plume becomes sourcelike and the present model applies is not precisely defined. One criterion is suggested by the numerical calculations for the mass flux on the axis which follows an inverse square law

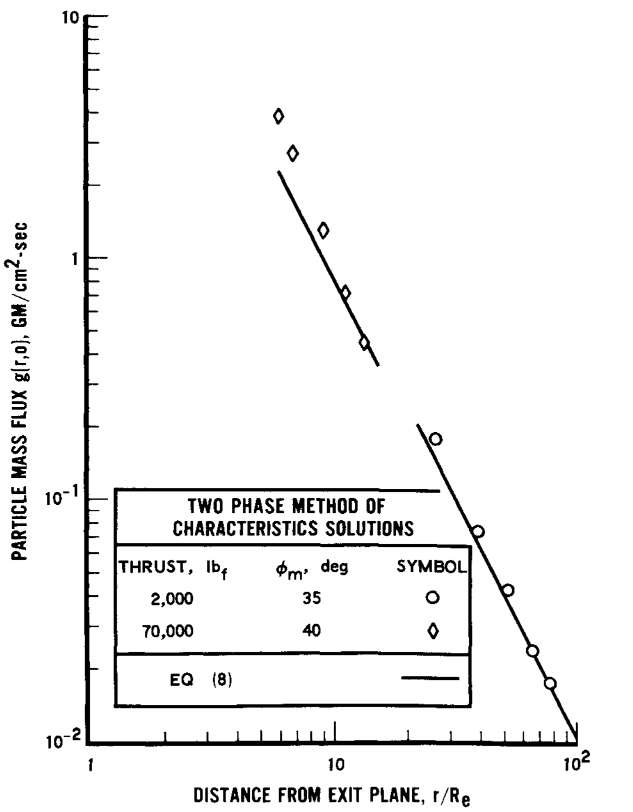

Fig. 4 Particle mass flux on the plume axis.

† L'Hospital's rule must be used to evaluate $\lim_{\xi \to 0} h(\xi)$ and c is chosen so that $h(0) = 1$.

dependence for $r/R_e \geqq 10$ (Fig. 4). An additional criterion is suggested by the plume length beyond which the particle and gas dynamics become uncoupled. This is essentially the condition after the particles pass into free molecule flow. On the plume axis, and for the largest particles, free molecule flow prevails for distances beyond about 7 exit radii downstream for the several plumes examined.

Since the present model deals only with the mass flux of particles, it contains no information about particle sizes or particle velocities. When this distribution is valid the particles have reached what is essentially a limiting velocity, which is less than the gas limiting velocity and is different for different particle sizes. The mass flux distribution is therefore directly applicable only to contamination estimates because heating and pressure estimates require an additional specification of velocity.

Finally, it is recognized that the particular form chosen for the fractional mass flux is not unique and there may be others which might improve the comparison. For example, the error function profile does not satisfy the conditions $w(1) = 1$ and $h(1) = 0$ identically. A function which satisfies these conditions and gives a reasonable fit to the numerical solutions is $w(\xi) = \sin^2(\xi\pi/2)$. However, this choice did not yield as satisfactory a fit to $h(\xi)$ for $\xi > 0.5$ as that given by Eq. (7).

The most important conclusion of this work is that the particle plume is accurately described as a spherical source flow. The particular form of the mass flux distribution given here is based on a set of calculations for conical nozzles and may not be uniformly satisfactory for other motors. However, the calculations against which these results were checked are representative of motors which are in common use which suggests that the simple particle plume mass flux model proposed here will be adequate for a majority of the cases of interest.

References

[1] Kliegel, J. R. and Nickerson, G. R., "Axisymmetric Two-Phase Perfect Gas Performance Program," Rept. 2874-6006-R000, April 1967, TRW Systems, Redondo Beach, Calif.

[2] Hoffman, R. J. et al., "Plume Contamination Effects Prediction," AFRPL-TR-71-109, Dec. 1971, McDonnell Douglas Astronautics Co., Huntington Beach, Calif.

[3] Hill, J. A. F. and Draper, J. S., "Analytical Approximation for the Flow from a Nozzle into a Vacuum," *Journal of Spacecraft and Rockets*, Vol. 3, No. 10, Oct. 1966, pp. 1552–1554.

[4] Pearce, B. E. and Wang, H. E., "Utilization of Solid Rocket Motors for Space Shuttle, Vol. VII, Staging Motor Exhaust Plume Effects on Shuttle Vehicle," Rept. ATR-73(7313-02)-1, Vol. VII, Aug. 1972, The Aerospace Corp., El Segundo, Calif.

Spin Stability of Torque-Free Systems Containing Rotors

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THE method described recently[1] for generating stability criteria applicable to spinning motions of certain torque-free systems cannot be used directly for systems containing free or driven rotors, for the development of the method involves the requirements that there be neither cyclic coordinates nor time-dependent constraints. Thus, dual-spin vehicles, satellites containing gyroscopic nutation dampers, etc., are excluded from consideration in Ref. 1. It is the purpose of this Note to indicate how such "gyroscopic" systems can be brought under purview of an extended form of the theory set forth in Ref. 1.

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All symbols not defined in what follows have the same meaning as in Ref. 1. A development closely analogous to that described in Ref. 1, and based on the principles discussed in Ref. 2, can be used to establish the validity of the results to be stated.

The system $\bar{S}$ to be examined consists of the system S considered in Ref. 1 together with N axisymmetric rotors R_ρ $(\rho = 1, \ldots, N)$. The rotors are attached to S in such a way that the mass center and axis of rotation of R_ρ remain fixed in reference from A, and R_ρ rotates with constant angular speed σ_ρ relative to A, whenever $\bar{S}$ performs a quasi-gyrostatic motion, that is, a motion during which the internal configuration of S remains unaltered. Such a motion is called a simple spin if ω has a constant magnitude and is at all times parallel to a principal axis of inertia of $\bar{S}$ for the mass center $\bar{S}^*$ of $\bar{S}$.

To formulate stability conditions for a motion of simple spin of $\bar{S}$, one may proceed as follows:

a) Select coordinates $q_1, \ldots, q_n$ governing the internal configuration of S in such a way that $q_1 = \cdots = q_n = 0$ during the simple spin to be investigated, and form V and D_r $(r = 1, \ldots, n)$ such that the requirements imposed by Eqs. (2–4) of Ref. 1 are satisfied.

b) Let Z_1, Z_2, Z_3 be a set of mutually perpendicular axes intersecting at $\bar{S}^*$, parallel to principal axes of inertia of $\bar{S}$ for $\bar{S}^*$ when $q_1 = \cdots = q_n = 0$, and numbered such that Z_1 plays the role of spin axis; let $\mathbf{z}_i$ be a unit vector pointing in the positive direction of Z_i $(i = 1, 2, 3)$; and let Z be a reference frame in which Z_1, Z_2, Z_3 are fixed.

c) Form I_{jk} (omit I_{23}) and $\mathbf{h}$ in accordance with

$$I_{jk} = \mathbf{z}_j \cdot \mathbf{I} \cdot \mathbf{z}_k \cdot (j, k = 1, 2, 3)$$

$$\mathbf{h} = \sum_{\rho=1}^{n} \sigma_\rho J_\rho \mathbf{u}_\rho$$

where $\mathbf{I}$ is the inertia dyadic of $\bar{S}$ for $\bar{S}^*$, J_ρ is the axial moment of inertia of R_ρ, and $\mathbf{u}_\rho$ is a unit vector parallel to the symmetry axis of R_ρ and having the same direction as the angular velocity of R_ρ relative to Z whenever $\bar{S}$ is performing a quasi-gyrostatic motion. Evaluate $\tilde{I}_{jj}$, $\tilde{I}_{1j,r}$, $\tilde{I}_{11,rs}$, $\tilde{V}_{,r}$, $\tilde{V}_{,rs}$, $\tilde{\mathbf{h}}$, $\tilde{\mathbf{h}}_{,r}$, and $\tilde{\mathbf{h}}_{,rs}$ $(j = 1, 2, 3; r, s = 1, \ldots, n)$, where the tilde denotes evaluation at $q_1 = \cdots = q_n = 0$ and a subscript comma followed by one or more letters signifies partial differentiation with respect to internal variables, these differentiations being performed in reference frame Z in the case of $\mathbf{h}$.

d) Form $a_{1j,r}$, $a_{11,rs}$, $\tilde{h}_j$, $\tilde{h}_{j,r}$, $\tilde{h}_{1,rs}$, and $\tilde{I}_{,rs}$ in accordance with

$$a_{11,r} = 0, \quad a_{1j,r} = \tilde{I}_{1j,r}/(\tilde{I}_{11} - \tilde{I}_{jj}) \quad (j = 2, 3)$$

$$a_{11,rs} = a_{12,r}a_{12,s} + a_{13,r}a_{13,s} \quad (r, s = 1, \ldots, n)$$

$$\tilde{h}_j = \mathbf{h} \cdot \mathbf{z}_j, \quad \tilde{h}_{j,r} = \tilde{\mathbf{h}}_{,r} \cdot \mathbf{z}_j - \tilde{h}_1 a_{ij,r} \quad (j = 1, 2, 3; r = 1, \ldots, n)$$

$$\tilde{h}_{1,rs} = \tilde{\mathbf{h}}_{,rs} \cdot \mathbf{z}_1 + \tilde{\mathbf{h}}_{,r} \cdot (a_{12,s}\mathbf{z}_2 + a_{13,s}\mathbf{z}_3) + \tilde{h}_1 a_{11,rs} + \tilde{\mathbf{h}}_{,s} \cdot (a_{12,r}\mathbf{z}_2 + a_{13,r}\mathbf{z}_3) \quad (r, s = 1, \ldots, n)$$

e) Verify that the simple spin under consideration can, in fact, occur, by ascertaining that, in accordance with "external" equations of motion

$$\tilde{h}_2 = \tilde{h}_3 = 0$$

whereas to satisfy "internal" equations of motion

$$\tilde{V}_{,r} - 2^{-1}(H - \tilde{h}_1)^2 \tilde{I}_{11,r}/\tilde{I}_{11}{}^2 - (H - \tilde{h}_1)\tilde{h}_{1,r}/\tilde{I}_{11} = 0 \quad (r = 1, \ldots, n)$$

where H is the magnitude of the angular momentum of $\bar{S}$ with respect to $\bar{S}^*$ in N.

f) Form α_2, α_3, β_{2r}, β_{3r}, and γ_{rs} in accordance with

$$\alpha_2 = H(\tilde{h}_1 - H)/\tilde{I}_{11} + H^2/\tilde{I}_{33}, \quad \alpha_3 = H(\tilde{h}_1 - H)/\tilde{I}_{11} + H^2/\tilde{I}_{22}$$

$$\beta_{2r} = -H\tilde{h}_{3,r}/\tilde{I}_{33}, \quad \beta_{3r} = H\tilde{h}_{2,r}/\tilde{I}_{22} \quad (r = 1, \ldots, n)$$

$$\gamma_{rs} = \tilde{V}_{,rs} + \tilde{h}_{1,rr}\tilde{h}_{1,s}/\tilde{I}_{11} + \tilde{h}_{2,r}\tilde{h}_{2,s}/\tilde{I}_{22} + \tilde{h}_{3,r}\tilde{h}_{3,s}/\tilde{I}_{33} + (H - \tilde{h}_1)[(\tilde{I}_{11,r}\tilde{h}_{1,s} + \tilde{I}_{11,s}\tilde{h}_{1,r})/\tilde{I}_{11} - \tilde{h}_{1,rs}]/\tilde{I}_{11} - (H - \tilde{h}_1)^2(\tilde{I}_{11}{}^{-2}\tilde{I}_{11,rs} - \tilde{I}_{11}{}^{-3}\tilde{I}_{11,r}\tilde{I}_{11,s})/2 \quad (r, s = 1, \ldots, n)$$